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PERTURBATIONS OF REGULARIZING MAXIMAL MONOTONE OPERATORS

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ABSTRACT

We consider $u'(t)+Au(t)\ni f(t)$, where A is maximal monotone in a Hilbert space H. Assume A is continuous or $A=\partial\varphi$ or $\operatorname{int} D(A)\neq\emptyset$ or $\dim H<\infty$. For suitably bounded f's, it is shown that the solution map $f\mapsto u$ is continuous, even if the f's are topologized much more weakly than usual. As a corollary we obtain the existence of solutions of $u'(t)+Au(t)\ni B(u(t))$, where B is a compact mapping in H.

0. Introduction

This paper is concerned with continuous dependence for the initial value problem

$$(0.1)\qquad \begin{cases} u'(t)+Au(t)\ni f(t) & (0\leqq t\leqq T), \\ u(0)=\text{given}, \end{cases}$$

where A is a maximal monotone operator in a Hilbert space H. Existence of solutions of (0.1) is briefly reviewed in Section 1. (Some of the general remarks in this introduction, and the results developed in this paper for continuous operators A, are applicable in the more general context of m-accretive operators A in an arbitrary Banach space. See [10], [12] and the remarks in Section 5. For brevity, however, in this paper we shall only consider Hilbert spaces.)

Holding $u(0)\in\overline{D(A)}$ fixed, how does the solution u of (0.1) depend on f? The standard result (see [2], [4], [8], for instance) is that the mapping $f\mapsto u$ is nonexpansive from $L^1([0,T];H)$ into $C([0,T];H)$. That is,

$$(0.2)\qquad |u_1(t)-u_2(t)|\leqq\int_0^t |f_1(\sigma)-f_2(\sigma)|\,d\sigma$$

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for $0 \leqq t \leqq T$. But in certain contexts, a sharper result is possible: Define

$$\text{(0.3)} \qquad |||f||| = \max_{0 \leq a \leq b \leq T} \left| \int_a^b f(\sigma) d\sigma \right| .$$

This is a norm on $L^1([0,T];H)$, substantially weaker than the usual one. Let $\int([0,T];H)$ be the vector space $L^1([0,T];H)$ retopologized, using the norm $|||\ |||$. For certain operators A, the mapping $f \mapsto u$ is continuous from suitable subsets of $\int([0,T];H)$ into $C([0,T];H)$. Results of this sort are a variant of an idea originally due to Gihman [6]. Some other continuous dependence results of Gihman type can be found in [10], [12] and in references cited therein. The present paper gives further examples of operators A with this property. Let $\|\ \|$ be the usual norm of $L^2([0,T];H)$. We shall show that if A is maximal monotone and either A is continuous or $A = \partial\varphi$ or int $D(A) \neq \emptyset$ or $\dim H < \infty$, then the mapping $f \mapsto u$ is continuous from $\|\ \|$-bounded subsets of $\int([0,T];H)$ into $C([0,T];H)$.

The topology given by $|||\ |||$ is weak enough that it has many compact sets. Indeed,

$$\text{(0.4)} \qquad \begin{gathered} \text{if } K \text{ is a compact convex subset of } H, \text{ then} \\ \{f : \text{Range}(f) \subseteq K\} \text{ is a compact convex subset of } \textstyle\int([0,T];H); \end{gathered}$$

see lemma 2 in [11]. Hence there does not seem to be much room for further weakening, since a compact topology is minimal among the Hausdorff topologies on a set.

Because $|||\ |||$ yields so many compact sets, our continuous dependence result can be used in fixed-point arguments. In this manner we prove the existence of solutions of

$$\text{(0.5)} \qquad u'(t) + Au(t) \ni B(u(t)),$$

where B is a compact operator in H. We remark that the theory of maximal monotone (or more generally, m-accretive) operators A and the theory of compact operators B have developed separately in the literature, and use very different techniques. Problems such as (0.5), involving the sum or difference of a monotone operator and a compact operator, have only been analyzed successfully in certain special cases [1], [7], [11], [12], and not in general. Additional special cases are given in the present paper.

In Section 1, below, we briefly review some results already known about maximal monotone operators and the initial value problem (0.1). This includes some specialized estimates which are applicable when $A = \partial\varphi$ or $\operatorname{int} D(A) \neq \emptyset$ or $\dim H < \infty$. In Section 2 those estimates are replaced with a single, weaker estimate. In Section 3 we prove some results on the convergence of integrals. One of those estimates is used in Section 4 to prove a theorem on the continuous dependence of u on f in (0.1). This theorem is quite sharp in several respects, as is shown by the examples in Section 5. The corollary in Section 6 says that (0.5) has a solution.

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1. Review of maximal monotone operators

The results stated in this section are proved in [2], [3], [4], [8]. Also given in those sources are numerous applications to nonlinear partial differential equations.

Throughout this paper, H will denote a real Hilbert space with norm $|\ \ |$ and inner product $(\ ,\ )$. Then $L^2([0,T];H)$ is also a Hilbert space, with norm $\|f\| = \{\int_0^T |f(\sigma)|^2 d\sigma\}^{1/2}$ and inner product $((f,g)) = \int_0^T (f(\sigma), g(\sigma))d\sigma$.

We consider multivalued mappings $A : H \to \{\text{subsets of } H\}$. Such a mapping has effective domain $D(A) = \{x \in H : Ax \neq \emptyset\}$ and effective range $R(A) = \bigcup_{x \in H} Ax$. Its inverse is the mapping $A^{-1} : H \to \{\text{subsets of } H\}$ defined by $A^{-1}y = \{x \in H : y \in Ax\}$. For simplicity of notation we identify A with its graph; thus $[x,y] \in A$ if and only if $y \in Ax$.

A multivalued mapping $A \subseteq H \times H$ is *monotone* if $(x_1 - x_2, y_1 - y_2) \geqq 0$ for all pairs $[x_1, y_1], [x_2, y_2] \in A$. It is *maximal monotone* if moreover it is not contained in any other monotone subset of $H \times H$.

Let A be maximal monotone. A *strong solution* of (0.1) is a continuous function $u : [0,T] \to \overline{D(A)}$ with the given value of $u(0)$, such that u is absolutely continuous on compact subsets of $(0,T]$ and, for almost every t in $[0,T]$, $u'(t)$ exists and lies in $f(t) - Au(t)$.

A *weak solution*, or *integral solution*, of (0.1) is a continuous function $u : [0,T] \to \overline{D(A)}$, with the given value of $u(0)$, such that

$$|u(t) - x|^2 \leqq |u(r) - x|^2 + 2\int_r^t (f(\sigma) - y, u(\sigma) - x)d\sigma \tag{1.1}$$

for all $[x,y] \in A$ and $0 \leqq r \leqq t \leqq T$.

For each $u(0) \in \overline{D(A)}$ and $f \in L^1([0,T];H)$ there is a unique weak solution u

of (0.1). For fixed $u(0)\in\overline{D(A)}$, the weak solutions u_1 and u_2 corresponding to forcing terms f_1 and f_2 satisfy (0.2), and also

$$|u_1(t)-u_2(t)|^2\leqq 2\int_0^t (f_1(\sigma)-f_2(\sigma),u_1(\sigma)-u_2(\sigma))d\sigma \tag{1.2}$$

for $0\leqq t\leqq T$.

Every strong solution is a weak solution. Conversely, certain conditions are known to imply that a weak solution is a strong solution. For instance, this is the case if f has bounded variation and $u(0)\in D(A)$. Since functions of bounded variation are dense in $L^1([0,T];H)$, the weak solutions are a natural generalization of strong solutions.

If the operator A satisfies certain additional hypotheses, then the weak solution u of (0.1) will have some additional regularity property, beyond just being continuous. Then we say A is *regularizing*. For instance, if $D(A)$ has nonempty interior, then for every $u(0)\in\overline{D(A)}$ and $f\in L^1([0,T];H)$ the solution u has bounded variation. In fact, we have this estimate:

$$\operatorname{Var}(u;[0,T])\leqq\frac{1}{2\rho}\left\{\rho+|u(0)-v_0|+TM+\int_0^T|f(\sigma)|d\sigma\right\}^2. \tag{1.3}$$

(See inequality (42) on page 80 of [4].) Here v_0 is any element of $\operatorname{int}D(A)$, and M and ρ are positive constants which may depend on A and v_0 but not on f or $u(0)$.

The same conclusions hold if A is any maximal monotone operator in a finite dimensional Hilbert space H. We shall sketch the proof, the main ingredients of which are given in [4]. By translation, we may assume $[0,0]\in A$. (This may affect the values of M, v_0, and ρ, but is otherwise without loss of generality.) Let H_0 be the span of $D(A)$, and let P be the orthogonal projection onto H_0. Let $A_0=A\cap(H_0\times H_0)$; then $D(A_0)=D(A)$. Since A is maximal monotone in H, it follows that $[x,y]\in A\Rightarrow[x,Py]\in A_0$, and that A_0 is maximal monotone in the Hilbert space H_0.

By results on pages 32–33 of [4], the domain of A_0 has nonempty interior in H_0. Now let u be the weak solution of (0.1). Then $u(\sigma)$ takes values in $\overline{D(A)}\subseteq H_0$, so for all $[x,y]\in A_0\subseteq A$ we have

$$\begin{aligned}|u(t)-x|^2-|u(r)-x|^2&\leqq 2\int_r^t(f(\sigma)-y,u(\sigma)-x))d\sigma\\&=2\int_r^t(Pf(\sigma)-y,u(\sigma)-x)d\sigma,\end{aligned}$$

for $0 \leqq r \leqq t \leqq T$. Thus u is also the solution, in H_0, of the initial value problem

$$\begin{cases} u'(t) + A_0 u(t) \ni Pf(t) & (0 \leqq t \leqq T), \\ u(0) = \text{given}. \end{cases}$$

Since $|Pf(t)| \leqq |f(t)|$, inequality (1.3) follows.

For another example of regularization, let $\varphi : H \to (-\infty, +\infty]$ be a convex, lower semicontinuous function. Also assume that φ is *proper*, i.e. that its effective domain $D(\varphi) = \{x \in H : \varphi(x) < +\infty\}$ is nonempty. Then the *subdifferential* of φ, defined by

$$\partial\varphi = \{[x, z] \in H \times H : \varphi(w) - \varphi(x) \geqq (z, w - x) \text{ for all } w \in H\},$$

is a maximal monotone operator in H. In the remainder of this paper we shall abbreviate these assumptions about an operator A by simply writing $A = \partial\varphi$. Many nonlinear parabolic differential equations can be written in the form (0.1) with $A = \partial\varphi$; see [2], [3], [4], [8] for examples.

The operator $\partial\varphi$ is regularizing. If $A = \partial\varphi$ and $f \in L^2([0, T]; H)$ and $u(0) \in \overline{D(A)}$, then the weak solution of (0.1) is a strong solution. Moreover, it satisfies this estimate: for any $z \in R(\partial\varphi)$,

$$\left\{\int_0^T \sigma\,|u'(\sigma)|^2 d\sigma\right\}^{1/2} \leqq \left\{\int_0^T \sigma\,|f(\sigma) - z\,|^2 d\sigma\right\}^{1/2} + \frac{1}{\sqrt{2}}\int_0^T |f(\sigma) - z\,|\,d\sigma$$
(1.4)
$$+ \frac{1}{\sqrt{2}}\,\mathrm{dist}(u(0), (\partial\varphi)^{-1}(z)).$$

This inequality is given in [2], [3], [4], [8], in the particular case where $\min\{\varphi(x) : x \in H\} = 0$ and $z = 0$. To reduce the problem to that case, we proceed as indicated in [2], [4]: Fix any $y \in (\partial\varphi)^{-1}(z)$. Define $\tilde{\varphi} : H \to (-\infty, +\infty]$ by $\tilde{\varphi}(w) = \varphi(w) - \varphi(y) - (z, w - y)$. Then $\min\{\tilde{\varphi}(x) : x \in H\} = 0$ and $0 \in R(\partial\tilde{\varphi})$; and u is the solution of $u'(t) + (\partial\tilde{\varphi})(u(t)) \ni f(t) - z$. Then (1.4) follows.

Still another class of regularizing maximal monotone operators A are those for which

(1.5) A is single-valued and continuous, and $D(A)$ is closed.

By "single-valued" we mean that for each $x \in H$, Ax contains at most one point. If (1.5) holds, $u(0) \in D(A)$, and $f \in L^2([0, T]; H)$, then the weak solution of (0.1) is a strong solution; and moreover it is continuously differentiable on $[0, T]$.

A more general class of initial value problems in an arbitrary Banach space is studied in [10].

2. Reduction to a single estimate

The remainder of this paper will be devoted to some consequences of (1.3) and (1.4). To avoid repetition we shall replace $\mathrm{Var}(u;[0,T])$ and $\int_0^T \sigma\,|u'(\sigma)|^2 d\sigma$ by a single, more complicated quantity which is in some sense dominated by both of those expressions. We shall hold $T>0$ fixed. For $0<a<T$ and positive integers n let $\theta_j=\theta_j^n(a)=a+(j/n)(T-a)$ $(j=0,1,2,\cdots,n)$. Thus $a=\theta_0<\theta_1<\cdots<\theta_n=T$, and $\theta_j-\theta_{j-1}=(T-a)/n$ for all j. For $u\in C([0,T];H)$ let

$$Q(u)=\sup_{a\in(0,T)}\ \sup_{n=1,2,\cdots}\ na\sum_{j=1}^{n}\int_{\theta_{j-1}^n(a)}^{\theta_j^n(a)}\!\!\!\!\!\!\!\!\!\!\!\!-\quad |u(\sigma)-u(\theta_{j-1}^n(a))|^2 d\sigma. \tag{2.1}$$

Lemma 1. *Let $u\in C([0,T];H)$ have bounded variation. Then*

$$Q(u)\leqq\tfrac12 T^2\{|u(0)|+\mathrm{Var}(u;[0,T])\}^2. \tag{2.2}$$

Proof. Fix any $a\in(0,T)$ and any positive integer n. Let $h=(T-a)/n$. Then

$$\begin{aligned}
na\sum_{j=1}^{n}\int_{\theta_{j-1}^n(a)}^{\theta_j^n(a)}|u(\sigma)-u(\theta_{j-1}^n(a))|^2 d\sigma &= na\int_0^h\sum_{j=1}^{n}|u(\theta_{j-1}^n(a)+s)-u(\theta_{j-1}^n(a))|^2 ds\\
&\leqq na\int_0^h 2\|u\|_{\sup}\mathrm{Var}(u;[0,T])ds\\
&=2\|u\|_{\sup}a(T-a)\mathrm{Var}(u;[0,T])\\
&\leqq\tfrac12 T^2\{|u(0)|+\mathrm{Var}(u;[0,T])\}^2.
\end{aligned}$$

Lemma 2. *Let $u\in C([0,T];H)$. Suppose u is absolutely continuous on compact subsets of $(0,T]$. Then*

$$Q(u)\leqq T\int_0^T\sigma\,|u'(\sigma)|^2 d\sigma. \tag{2.3}$$

Proof. Fix any $a\in(0,T)$ and positive integer n. Let $\theta_j=\theta_j^n(a)$. Then

$$\begin{aligned}
\int_{\theta_{j-1}}^{\theta_j}|u(\sigma)-u(\theta_{j-1})|^2 d\sigma &= \int_{\theta_{j-1}}^{\theta_j}\left|\int_{\theta_{j-1}}^{\sigma}u'(s)ds\right|^2 d\sigma\leqq\int_{\theta_{j-1}}^{\theta_j}\int_{\theta_{j-1}}^{\sigma}|u'(s)|^2 ds\,d\sigma\\
&=\int_{\theta_{j-1}}^{\theta_j}(\theta_j-s)|u'(s)|^2 ds\leqq\frac{T-a}{na}\int_{\theta_{j-1}}^{\theta_j}s\,|u'(s)|^2 ds.
\end{aligned}$$

Now (2.3) follows immediately.

REMARK. From inequalities (1.3) through (2.3) it follows that if $A = \partial\varphi$ or $\operatorname{int} D(A) \neq \emptyset$ or $\dim H < \infty$, and f is permitted to vary in a bounded subset of $L^2([0,T];H)$, and u is the solution of (0.1) with $u(0)$ held fixed, then $Q(u)$ remains bounded.

It is not known whether a similar estimate can be proved when $A = A_1 + \partial\varphi$, with A_1 maximal monotone and $\operatorname{int} D(A_1) \cap D(\partial\varphi) \neq \emptyset$, although in that case A is known to be maximal monotone [4].

3. Convergence of integrals

Recall that a sequence or net $\{f_n\}$ converges *weakly* in $L^2([0,T];H)$ to a limit f if $((f_n - f, g)) \to 0$ for every $g \in L^2([0,T];H)$. Hereafter this convergence will be denoted by $f_n \xrightarrow{w} f$.

This section gives some consequences of $|||\ \ |||$-convergence, where $|||\ \ |||$ is defined as in (0.3). The $|||\ \ |||$-convergence is similar to weak convergence, as is shown by the proposition below. However, in the study of continuous dependence results such as those in Section 4, $|||\ \ |||$-convergence is more appropriate than weak convergence; this is shown by Examples 1 and 2 in Section 5.

PROPOSITION. *Let $\{f_n\}$ be a sequence (or more generally, a net) in $L^2([0,T];H)$; and let $f \in L^2([0,T];H)$. Then:*

(a) *If $\sup_n \|f_n\| < \infty$ and $|||f_n - f||| \to 0$ then $f_n \xrightarrow{w} f$.*

(b) *If $\bigcup_n \mathrm{Range}(f_n)$ is a relatively compact subset of H, and $f_n \xrightarrow{w} f$, then $|||f_n - f||| \to 0$.*

PROOF OF (a). Suppose $g : [0,T] \to H$ is a step-function; i.e., g takes some constant value g_i on each subinterval (t_{i-1}, t_i) of some partition: $0 = t_0 < t_1 < \cdots < t_m = T$. Then as $n \to \infty$,

$$|((f_n - f, g))| \leqq \sum_{i=1}^{m} \left| \int_{t_{i-1}}^{t_i} [f_n(s) - f(s)]ds \right| |g_i| \leqq |||f_n - f||| \sum_{i=1}^{m} |g_i| \to 0.$$

Such step-functions are dense in $L^2([0,T];H)$; hence (a).

PROOF OF (b). Immediate from (a) and (0.4).

Suppose $f_n \xrightarrow{w} f_\infty$ and $u_n \xrightarrow{w} v$ in some Hilbert space with inner product $((\ ,\ ))$. In general, we cannot conclude that $((f_n, u_n)) \to ((f_\infty, v))$. But that conclusion does hold if one of the sequences $\{f_n\}$, $\{u_n\}$ lies in a compact subset of the Hilbert space. It also holds if each of the sequences $\{f_n\}$, $\{u_n\}$ satisfies some condition slightly weaker than compactness; then each of the sequences "compensates" for the other. An example of this principle of "compensated compactness" is given in the lemma below. Other examples can be found in [9], [13] and in papers cited therein.

LEMMA 3. *Let $\{f_\alpha\}$ be a net (i.e., a generalized sequence) in $L^2([0,T];H)$. Let $\{u_\alpha\}$ be a net in $C([0,T];H)$, indexed by the same directed set $\{\alpha\}$. Also let f_∞ and v be elements of $L^2([0,T];H)$. Assume $|||f_\alpha - f_\infty||| \to 0$ and $u_\alpha \xrightarrow{w} v$. Also assume that $c_1 = \sup_\alpha \|f_\alpha\|$, $c_2 = \sup_\alpha Q(u_\alpha)$, and $c_3 = \sup_\alpha \sup_\sigma |u_\alpha(\sigma)|$ are all finite. Then*

$$\int_a^b (f_\alpha(\sigma), u_\alpha(\sigma))d\sigma \to \int_a^b (f_\infty(\sigma), v(\sigma))d\sigma$$

as $\alpha \to \infty$, uniformly for all $a, b \in [0,T]$.

PROOF. For $0 \leqq r \leqq t \leqq T$, we have $|\int_r^t (f_\alpha(s), u_\alpha(s))ds| \leqq c_3 \int_r^t |f_\alpha(s)|ds \leqq c_1 c_3 \sqrt{t-r}$. This converges to 0 when $t - r \to 0$, uniformly for all α. Hence the functions $(a,b) \mapsto \int_a^b (f_\alpha, u_\alpha)$ are uniformly equicontinuous on $[0,T]\times[0,T]$. Hence it suffices to show $\int_a^b (f_\alpha, u_\alpha)$ converges to $\int_a^b (f_\infty, v)$ for fixed a, b; and we only need to consider a and b in a dense set. Thus we may fix a, b; and we may assume $0 < a < b \leqq T$. In fact, we may assume $b = T$; for then the general case can be recovered by subtracting the results obtained for two different values of a.

Let χ be the characteristic function of the interval $[a, T]$.

Temporarily fix some positive integer n. Define $\theta_j = \theta_j^n(a)$ as in Section 2. For $f = f_\alpha$ or $f = f_\infty$ we have

$$\begin{aligned}
&\left|\sum_{j=1}^n \int_{\theta_{j-1}}^{\theta_j} (f(\sigma), u_\alpha(\sigma) - u_\alpha(\theta_{j-1}))d\sigma\right| \\
&\leqq \sum_{j=1}^n \int_{\theta_{j-1}}^{\theta_j} \left[\frac{1}{\sqrt[4]{n}}|f(\sigma)|\right][\sqrt[4]{n}|u_\alpha(\sigma) - u_\alpha(\theta_{j-1})|]d\sigma \\
&\leqq \sum_{j=1}^n \int_{\theta_{j-1}}^{\theta_j} \left\{\frac{1}{2}\left[\frac{1}{\sqrt[4]{n}}|f(\sigma)|\right]^2 + \frac{1}{2}[\sqrt[4]{n}|u_\alpha(\sigma) - u_\alpha(\theta_{j-1})|]^2\right\} d\sigma \\
&\leqq \frac{1}{2\sqrt{n}}\left[\|f\|^2 + \frac{1}{a}Q(u_\alpha)\right].
\end{aligned}$$

Hence, by several applications of the triangle inequality,

$$\begin{aligned}
&\left|\int_a^T (f_\alpha(\sigma), u_\alpha(\sigma))d\sigma - \int_a^T (f_\infty(\sigma), v(\sigma))d\sigma\right| \\
&\leqq \frac{1}{2\sqrt{n}}\|f_\alpha\|^2 + \frac{1}{2\sqrt{n}}\|f_\infty\|^2 + \frac{1}{a\sqrt{n}}Q(u_\alpha)
\end{aligned}$$

$$+\left|\int_a^T (f_\infty(\sigma), u_\alpha(\sigma)-v(\sigma))d\sigma\right|$$

$$+\sum_{j=1}^{n} |u_\alpha(\theta_{j-1})|\left|\int_{\theta_{j-1}}^{\theta_j} [f_\alpha(\sigma)-f_\infty(\sigma)]d\sigma\right|$$

$$\leqq \frac{1}{2\sqrt{n}}\left[c_1^2+\|f_\infty\|^2+\frac{2c_2}{a}\right]$$

$$+|((\chi f_\infty, u_\alpha - v))| + nc_3 ||| f_\alpha - f_\infty |||.$$

Hold n fixed and take limits as $\alpha\to\infty$. We obtain

$$\limsup_{\alpha\to\infty}\left|\int_a^T (f_\alpha(\sigma), u_\alpha(\sigma))d\sigma - \int_a^T (f_\infty(\sigma), v(\sigma))d\sigma\right| \leqq \frac{1}{2\sqrt{n}}\left[c_1^2+\|f_\infty\|^2+\frac{2c_2}{a}\right].$$

Now let $n\to\infty$. This completes the proof.

4. Convergence of evolutions

THEOREM. *Let A be maximal monotone in H. Assume that* (1.5) *holds or* $\operatorname{int} D(A)\neq\varnothing$ *or* $\dim H<\infty$ *or* $A=\partial\varphi$. *Let* $u(0)\in\overline{D(A)}$. *Then the solution map* $f\mapsto u$ *for the initial value problem* (0.1) *is continuous from* $\|\ \|$*-bounded subsets of* $\int([0,T];H)$, *into* $C([0,T];H)$.

PROOF. Let $\{f_k\}$ be a bounded sequence in $L^2([0,T];H)$, and let $f_\infty\in L^2([0,T];H)$. Let $\{u_k\}$ and u_∞ be the corresponding solutions of (0.1). Assume that $|||f_k-f_\infty|||\to 0$. We are to show that $u_k\to u_\infty$ uniformly on $[0,T]$.

We first consider the case in which (1.5) holds. We shall apply proposition 6.5 of [10], with $D=D(A)$ and $A_p(s,x)=A(x)+f_p(s)$. All of the hypotheses of that proposition are immediately satisfied except (6.8). An inspection of the proof of that proposition shows that (6.8) can easily be replaced by the condition

$$\lim_{h\downarrow 0}\sup_{0\le b-a\le h}\sup_p\int_a^b \sup_{y\in K}|A_p(s,y)|ds=0$$

for compact sets $K\subseteq D$. This is clearly satisfied in the present context, since

$$\int_a^b |f_n(s)|ds \leqq \{b-a\}^{1/2}\left\{\int_a^b |f_n(s)|^2 ds\right\}^{1/2} \leqq \{b-a\}^{1/2}\sup_k\|f_k\|.$$

Hence $u_k\to u_\infty$ uniformly on $[0,T]$.

We now turn to the case where $\operatorname{int} D(A)\neq\varnothing$ or $\dim H<\infty$ or $A=\partial\varphi$. It suffices to show that some subsequence of $\{u_k\}$ converges uniformly to u_∞. For, if $\{\|u_k-u_\infty\|_{\sup}\}$ does not converge to 0, then it can be replaced with some subsequence which is bounded away from 0; and then no subsubsequence of that subsequence can converge to 0.

From (0.2) we have

$$|u_k(t)-u_\infty(t)| \leqq \int_0^T |f_k(\sigma)-f_\infty(\sigma)|\,d\sigma$$
$$\leqq T^{1/2}\left\{\left[\int_0^T |f_k(\sigma)|^2 d\sigma\right]^{1/2}+\left[\int_0^T |f_\infty(\sigma)|^2 d\sigma\right]^{1/2}\right\}.$$

Hence the u_k's are uniformly bounded. Passing to a subsequence, we may assume that $\{u_k\}$ converges weakly to some limit $v \in L^2([0,T];H)$.

From (1.2) we have

$$|u_j(t)-u_k(t)|^2 \leqq 2\int_0^t ((f_j(\sigma)-f_k(\sigma), u_j(\sigma)-u_k(\sigma))d\sigma.$$

The nets $\{f_j - f_k\}$ and $\{u_j - u_k\}$ satisfy the hypotheses of Lemma 3. Here the net $\{f_j - f_k\}$ is $|||\ \ |||$-converging to 0, and $(u_j - u_k)\xrightarrow{w} 0$. Hence the above integral converges to 0, uniformly for all t in $[0,T]$. Therefore $\{u_k\}$ is a Cauchy sequence in $C([0,T];H)$. Hence $u_k \to v$ uniformly on $[0,T]$. Therefore v is continuous, and $v(0)=u(0)$.

From (1.1) we have

$$|u_k(t)-x|^2 \leqq |u_k(r)-x|^2 + 2\int_r^t (f_k(\sigma)-y, u_k(\sigma)-x)d\sigma$$

for all $0 \leqq r \leqq t \leqq T$ and $[x,y]\in A$. Apply Lemma 3 again; we obtain

$$|v(t)-x|^2 \leqq |v(r)-x|^2 + 2\int_r^t (f_\infty(\sigma)-y, v(\sigma)-x)d\sigma.$$

Thus v is the weak solution of (0.1) for $f=f_\infty$. That is, $v=u_\infty$. This completes the proof.

5. Examples and remarks

The examples in this section show that the theorem of Section 4 is sharp in several respects.

EXAMPLE 1. The weak and $|||\ \ |||$-topologies on $L^2([0,T];H)$ are not comparable: Neither of their convergences implies the other.

For an example in which $|||f_n|||\to 0$ but $\{f_n\}$ does not converge weakly, let $H=R$, $T=2\pi$, and $f_n(t)=n^{1/2}\sin(nt)$. Then $|||f_n|||=2/\sqrt{n}\to 0$, but $\|f_n\|=\sqrt{n\pi}\to\infty$. Any weakly convergent sequence is bounded; so $\{f_n\}$ does not converge weakly.

For an example in which $f_n \xrightarrow{w} 0$ but $\{f_n\}$ does not $|||\ |||$-converge, take $H = L^2(R;R)$, and take any $T > 0$. Then by lemma III.11.16 of [5], $L^2([0,T];H) = L^2([0,T]\times R;R) = L^2(R;L^2([0,T];R))$. Let any $\xi_0 \in H$ other than 0 be given. Let $\xi_n(\theta) = \xi_0(\theta - n)$, for all $\theta \in R$. Then $\xi_n \in H$ with $|\xi_n| = |\xi_0| > 0$ for all n, so $\{\xi_n\}$ does not converge strongly to 0. It is easy to verify that $\{\xi_n\}$ converges to 0 in the weak topology of H. Now define $f_n \in L^2([0,T];H)$ by taking $f_n(t) = \xi_n$ for all t in $[0,T]$. Then $f_n \xrightarrow{w} 0$; hence $\sup_n \|f_n\| < \infty$. By part (a) of the proposition in Section 3, if $\{f_n\}$ is $|||\ |||$-convergent, then it is weakly convergent to the same limit; hence that limit must be 0. But $|||f_n||| = T|\xi_0| > 0$ for all n, so the sequence $\{f_n\}$ does not $|||\ |||$-converge.

EXAMPLE 2. A simple observation shows that $|||\ |||$-convergence is appropriate for continuous dependence results for (0.1). Let (1.5) hold, and let $u(0) \in D(A)$. Let $\{f_n\}$ and f_∞ be any elements of $L^1([0,T];H)$, and let $\{u_n\}$ and u_∞ be the corresponding solutions of (0.1). Then

$$\|u_n - u_\infty\|_{\sup} \to 0 \Rightarrow |||f_n - f_\infty||| \to 0. \tag{5.1}$$

PROOF By assumption, $u_n \to u_\infty$ uniformly on $[0,T]$. Since the u_n's are continuous, $K = \bigcup_{n=1}^{\infty} \mathrm{Range}(u_n)$ is relatively compact. Since A is continuous on $D(A)$, it is uniformly continuous on K. Hence $A \circ u_n \to A \circ u_\infty$ uniformly on $[0,T]$. Therefore

$$\int_a^b f_n(s)ds = u_n(b) - u_n(a) + \int_a^b A(u_n(s))ds$$

$$\to u_\infty(b) - u_\infty(a) + \int_a^b A(u_\infty(s))ds = \int_a^b f_\infty(s)ds$$

uniformly for all a, b in $[0,T]$.

EXAMPLE 3. Conclusion (5.1) does not hold for the other classes of operators A considered in Section 4. In fact, for those classes, the solution mapping $f \mapsto u$ need not be injective. For a simple example of this, let $H = R$. Let $A(0) = [-1,1]$, and let $A(x) = \mathrm{sign}(x)$ for $x \neq 0$. Then A is maximal monotone. We have $A = \partial\varphi$, where $\varphi(x) = |x|$ is convex, lower semicontinuous, and proper. Also $D(A) = H$ has nonempty interior, and $H = R$ has finite dimension. Now let $u(0) = 0$, and let f be any measurable map from $[0,T]$ into $[-1,1]$. There are many such f's, but for all of them the solution of (0.1) is $u = 0$.

EXAMPLE 4. The continuous dependence result proved in Section 4 for certain classes of A's does not hold for an arbitrary maximal monotone operator

A. It fails in the following example (which was given in [12] for a slightly different purpose). Let $H = L^2([-\pi, \pi]; R)$. Elements of H will be viewed as functions $x(\theta)$ defined for all $\theta \in R$, periodic in θ with period 2π. Let $A = \partial/\partial\theta$, with periodic boundary conditions. That is: $(Ax)(\theta) = x'(\theta)$, with $D(A) = \{x : x$ is absolutely continuous on $[-\pi, \pi]$, $x(-\pi) = x(\pi)$, and $x' \in H\}$. Then A is linear, densely-defined, and maximal monotone. A is skew-adjoint, and is the generator of a group of isometries e^{tA}. These are the translations $[e^{tA}x](\theta) = x(t+\theta)$.

For each positive integer n, define $f_n : [0, T] \to H$ by taking $f_n(t, \theta) = \sin(nt - n\theta)$ for $t \in [0, T]$, $\theta \in [-\pi, \pi]$. Let $f_\infty = 0$. The reader can verify that $\sup_n \|f_n\| < \infty$ and $|||f_n - f_\infty||| \to 0$.

Let $u(0) = u(0, \theta) = 0$ $(\theta \in [-\pi, \pi])$. Let $\{u_n\}$ and u_∞ be the solutions of (0.1) corresponding to forcing terms $\{f_n\}$ and f_∞. Then it is easy to show that $u_n(t, \theta) = t \sin(nt - n\theta)$ for $n = 1, 2, 3, \cdots$; but $u_\infty = 0$. Hence $\{u_n\}$ does not converge in $C([0, T]; H)$ to u_∞.

6. Compact perturbations

Finally, we shall apply our continuous dependence results to the initial value problem

$$\text{(6.1)} \qquad \begin{cases} u'(t) + Au(t) \ni B(u(t)) & (0 \leqq t \leqq T), \\ u(0) = \text{given}, \end{cases}$$

where B is a compact mapping. The illustrative corollary below is chosen for its simplicity, not its generality. It is not yet clear what is the broadest possible version of this theory. Results in [1] permit B to be time-dependent and set-valued, but impose some other restrictions not made here. Some other related results have also been obtained, by a different method, in [7].

It is not yet known whether conclusion (b), below, is valid for an arbitrary maximal monotone operator A. The analogous question for m-accretive operators in an arbitrary Banach space also is still open; see the introduction to [12].

COROLLARY. *Let A be a maximal monotone operator in a Hilbert space H. Assume* (1.5) *holds or $A = \partial\varphi$ or* $\operatorname{int} D(A) \neq \emptyset$ *or* $\dim H < \infty$. *Let* $u(0) \in \overline{D(A)}$, *and let $T > 0$. Then*:

(a) *Let K be a compact convex subset of H. Let* $F = \{f \in L^1([0, T]; H) : \text{Range}(f) \subseteq K\}$, *and let F be topologized as a subset of* $\int([0, T]; H)$. *For each $f \in F$ let u_f be the corresponding solution of* (0.1).

Then F is compact, and the mapping $f \mapsto u_f$ is continuous from F into $C([0,T];H)$. Hence $\{u_f : f \in F\}$ is a compact subset of $C([0,T];H)$; and so $\{u_f(T): f \in F\}$ and $\{u_f(t): f \in F,\ t \in [0,T]\}$ are compact subsets of H.

(b) *Let L be a compact subset of H. Let $B : \overline{D(A)} \to L$ be continuous. Then there exists at least one solution of* (6.1).

PROOF. Part (a) is immediate from (0.4) and the theorem in Section 4. For part (b), let K be the closed convex hull of L. Then K is compact and convex, and the results of part (a) are applicable. The mapping $v \mapsto B \circ v$ is easily seen to be continuous from $C([0,T];H)$ into F. Composing it with the mapping $f \mapsto u_f$, we obtain a continuous mapping from F into F. The set F is compact and convex. By the Schauder–Tychonoff Fixed Point Theorem [5], that composition has a fixed point. That is, there exists at least one $f \in F$ such that $B \circ u_f = f$. Then u_f is a solution of (6.1).

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